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Classical Poisson structures and *r*-matrices from constrained flows

Yunbo Zeng†‡ and Jarmo Hietarinta†

† Department of Physical Sciences, University of Turku, FIN-20500 Turku, Finland
‡ Department of Applied Mathematics,Tsinghua University, Beijing 100084, People's Republic of China§

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Abstract. We construct the classical Poisson structures and r-matrices for some finite-dimensional integrable Hamiltonian systems obtained by constraining the flows of soliton equations in a certain way. This approach allows us to produce new kinds of classical (dynamical) Yang–Baxter structures. To illustrate this method we present the r-matrices associated with the constrained flows of the Kaup–Newell, KdV, WKI and TG hierarchies, all generated by a two-dimensional eigenvalue problem. Some of the r-matrices thus obtained depend only on the spectral parameters, but others depend also on the dynamical variables. For consistency they have to obey a classical Yang–Baxter-type equation, possibly with dynamical extra terms.

1. Introduction

Integrable finite-dimensional systems that admit a classical r-matrix depending only on the spectral parameters have been studied extensively [1]. Recently it has been found that in many cases the corresponding r-matrix depends also on the dynamical variables [2–7]. For example, the celebrated Calogero–Moser system has been shown to possess a dynamical r-matrix [6, 7]. In contrast to the well-studied case of r-matrices depending only on spectral parameters, the general theory of dynamical r-matrices has not yet been established. New examples are therefore needed in the search for the underlying structure, and the method presented below seems to be quite useful for this purpose.

Let us first recall some basic results about the r-matrix approach to dynamical systems. In [2] it was pointed out that the integrability of a dynamical system system (along with many other useful properties) can be shown in a straightforward way (see section 2.3), if the Poisson structure can be written in the following form:

$$\{M^{(1)}(\alpha_1) \overset{\otimes}{,} M^{(2)}(\alpha_2)\} = [r^{(12)}(\alpha_1, \alpha_2), M^{(1)}(\alpha_1)] - [r^{(21)}(\alpha_2, \alpha_1), M^{(2)}(\alpha_2)]. \tag{1}$$

Here the symbol $\{\, . \overset{\otimes}{,} . \,\}$ has been introduced to handle Poisson brackets between matrices: if M and N are matrices then $\{M \overset{\otimes}{,} N\}^{kl}_{ij} := \{M^k_i, N^l_j\}$, where on the right-hand side the standard Poisson bracket is used. The square brackets on the right-hand side of (1) stand for matrix commutators, and the superscripts refer to the vector spaces on which the matrices act non-trivially: $M^{(1)}(\alpha_1) = M(\alpha_1) \otimes 1$ and $M^{(2)}(\alpha_2) = 1 \otimes M(\alpha_2)$. The equation itself is defined on $V_1 \otimes V_2$, where V_i are identical d-dimensional vector

§ Permanent address.

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spaces (in this paper $d = 2$), thus all matrices are $d^2 \times d^2$ dimensional. The spectral parameters α_i are associated with the vector spaces, so it is not necessary to write them out explicitly. Note also that the usual permutation matrix P permutes *only* the vector spaces: $r^{(21)}(\alpha_1, \alpha_2) = P^{(12)} r^{(12)}(\alpha_1, \alpha_2) P^{(12)}$.

For consistency the Poisson bracket above has to obey the Jacobi identity and this in turn implies an equation for the r-matrix. The Jacobi identity for M reads

$$\{M^{(1)} \overset{\otimes}{,} \{M^{(2)} \overset{\otimes}{,} M^{(3)}\}\} + \{M^{(2)} \overset{\otimes}{,} \{M^{(3)} \overset{\otimes}{,} M^{(1)}\}\}\{M^{(3)} \overset{\otimes}{,} \{M^{(1)} \overset{\otimes}{,} M^{(2)}\}\} = 0. \tag{2}$$

This equation is defined on $V_1 \otimes V_2 \otimes V_3$ so, e.g., $M^{(2)} = 1 \otimes M(\alpha_2) \otimes 1$. If we allow for the possibility that the r's depend on the dynamical variables, then direct application of (1) to (2) leads to the requirement

$$[R^{(123)}, M^{(1)}] + [R^{(231)}, M^{(2)}] + [R^{(312)}, M^{(3)}] = 0 \tag{3}$$

where

$$R^{(ijk)} := r^{(ijk)} + \{M^{(j)} \overset{\otimes}{,} r^{(ik)}\} - \{M^{(k)} \overset{\otimes}{,} r^{(ij)}\} \tag{4}$$

$$r^{(ijk)} := [r^{(ij)}, r^{(ik)}] + [r^{(ij)}, r^{(jk)}] + [r^{(kj)}, r^{(ik)}]. \tag{5}$$

If the $r^{(ij)}$'s do not depend on dynamical variables, then the Jacobi identity (3) should be satisfied by $r^{(ijk)} = 0$. This equation is *almost* the classical Yang–Baxter equation, which would be obtained if we had $r^{(kj)} = -r^{(jk)}$ (in which case the last term in (5) could be written as $[r^{(ik)}, r^{(jk)}]$). It turns out, however, that most of the examples presented here, e.g, (54), do not have such a antisymmetry, thus the index order in (5) is crucial.

For the dynamical r-matrices presented below one finds that $R^{(ijk)} \neq 0$. In [6] Sklyanin observed that in such a case the Jacobi identity (3) can nevertheless be satisfied, if

$$R^{(ijk)} = [X^{(ijk)}, M^{(j)}] - [X^{(kij)}, M^{(k)}] \tag{6}$$

for some matrix X. We call this equation the *dynamical, classical Yang–Baxter equation.* The special case $X^{(ijk)} = X^{(kij)}$ was used before in [5], but for the examples presented in section 3 the Jacobi identity is satisfied due to (6), where $X^{(ijk)} \neq X^{(kij)}$.

Once the Lax representation of a dynamical system is given, constructing the classical Poisson structure and the related r-matrix defined above is straightforward. Often the r-matrix depends only on the spectral parameters, but, as mentioned above, in some cases it turns out to depend on the dynamical variables as well. In the present paper, to illustrate the above, we describe the classical Poisson structures and the related classical Yang–Baxter equations of some dynamical systems obtained from constrained flows of soliton hierarchies. In section 2 we give a brief introduction to the theory behind this by discussing, as an example, the constrained flows for the Kaup–Newell hierarchy. The r-matrices are then presented in section 3, and we also discuss the way they satisfy the Jacobi identity. This is first done for the constrained flows of Kaup–Newell hierarchy of section 2, and then for the KdV, G Tu (TG) and the Wadati–Konno–Ichikawa (WKI) hierarchy. Our main results are the dynamical r-matrices connected with the last two cases. Since large families of constrained flows can be reduced from various soliton hierarchies, the other purpose of this paper is to emphasize that constrained flows may provide a useful way for searching many types of r-matrices.

2. Integrable constrained flows

To make the paper self contained we first briefly describe how finite-dimensional integrable systems and their Lax representation can be constructed from constrained flows of soliton equations. We will use the Kaup–Newell (KN) hierarchy as an illustration; for further details, see [8–11].

2.1. The hierarchy of Hamiltonian flows

Let us start by considering the Kaup–Newell eigenvalue problem [12]:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(u,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad U(u,\lambda) = \begin{pmatrix} -\lambda^2 & \lambda q \\ \lambda r & \lambda^2 \end{pmatrix}. \tag{7}$$

(Here and in what follows we denote $u^t = (q, r)$.) First, we solve the adjoint representation of (7) [13, 14]

$$V_x = [U, V] \equiv UV - VU \tag{8}$$

where V has a Laurent series expansion

$$V(u,\lambda) = \sum_{m=0}^{\infty}\begin{pmatrix} a_m(u) & b_m(u) \\ c_m(u) & -a_m(u) \end{pmatrix}\lambda^{-m}. \tag{9}$$

Equations (8) and (9) lead to the recursion relations

$$\begin{aligned} b_{m+2} &= -qa_{m+1} - \tfrac{1}{2}b_{m,x} \\ c_{m+2} &= -ra_{m+1} + \tfrac{1}{2}c_{m,x} \\ a_m &= \tfrac{1}{2}\partial_x^{-1}(qc_{m-1,x} + rb_{m-1,x}) \end{aligned} \tag{10}$$

and to the parity constraints $a_{2m+1} = b_{2m} = c_{2m} = 0$. The first few terms are as follows:

$$\begin{aligned} &a_0 = 1, \qquad a_2 = -\tfrac{1}{2}qr, \qquad a_4 = \tfrac{3}{8}q^2r^2 + \tfrac{1}{4}(rq_x - qr_x), \ldots \\ &b_1 = -q, \qquad b_3 = \tfrac{1}{2}(q^2r + q_x), \ldots \\ &c_1 = -r, \qquad c_3 = \tfrac{1}{2}(r^2q - r_x), \ldots . \end{aligned} \tag{11}$$

The recursion relation (10) can also be expressed as

$$\begin{pmatrix} c_{2m+1} \\ b_{2m+1} \end{pmatrix} = L\begin{pmatrix} c_{2m-1} \\ b_{2m-1} \end{pmatrix} \qquad L = \frac{1}{2}\begin{pmatrix} \partial_x - r\partial_x^{-1}q\partial_x & -r\partial_x^{-1}r\partial_x \\ -q\partial_x^{-1}q\partial_x & -\partial_x - q\partial_x^{-1}r\partial_x \end{pmatrix}. \tag{12}$$

Next let us consider a 'truncation' of the expression (9):

$$V^{(n)}(u,\lambda) \equiv (\lambda^{2n}V)_+ \equiv \sum_{m=0}^{n-1}\begin{pmatrix} a_{2m}\lambda^{2n-2m} & b_{2m+1}\lambda^{2n-2m-1} \\ c_{2m+1}\lambda^{2n-2m-1} & -a_{2m}\lambda^{2n-2m} \end{pmatrix} \tag{13}$$

and using it let us define the nth flow of the eigenfunction by

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_n} = V^{(n)}(u,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{14}$$

Then the compatibility condition of (7) and (14) gives rise to a zero-curvature representation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0 \qquad n = 1, 2, \ldots . \tag{15}$$

Due to the construction of $V^{(n)}$ in (13) only terms of lowest order in λ contribute, yielding the KN hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\begin{pmatrix} c_{2n-1} \\ b_{2n-1} \end{pmatrix} = J\frac{\delta H_{2n-2}}{\delta u} \qquad J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{16}$$

where

$$H_{2m} = \frac{1}{2m}(4a_{2m+2} - rb_{2m+1} - qc_{2m+1}) \qquad H_0 = -qr. \tag{17}$$

In the above construction all other steps are straightforward, except the fact that the flow (16) can be written in terms of a Hamiltonian H_{2n-2}, and that the Hamiltonians so obtained are in involution with respect to the ordinary infinite-dimensional Poisson bracket [14]. (Also in some cases one has to add a lowest order in λ correction term to $V^{(n)}$.) One elegant method to derive this is by using certain trace identities [13].

2.2. *The constrained flow*

In order to construct a finite-dimensional integrable system we will take N copies of (7) with distinct λ_j's

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = U(u, \lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix} \qquad j = 1, \ldots, N \tag{18}$$

and these ψ's will be the new dynamical variables (although sometimes there will be others as well). The additional ingredient we need is a constraint that relates u to the ψ's. Furthermore this constraint must be such that it preserves the integrability of the original system, i.e. it must be invariant under the flows (16).

One suitable constraint is obtained as follows [8]. It is known [15] that for systems (18) with $\mathrm{Tr}(U) = 0$ we have (up to a constant factor)

$$\frac{\delta\lambda}{\delta u_i} = \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} \psi_1\psi_2 & -\psi_1^2 \\ \psi_2^2 & -\psi_1\psi_2 \end{pmatrix}\frac{\partial U(u,\lambda)}{\partial u_i}\right] \tag{19}$$

which in the present case implies

$$\frac{\delta\lambda}{\delta u} = \frac{1}{2}\begin{pmatrix} \lambda\psi_2^2 \\ -\lambda\psi_1^2 \end{pmatrix}. \tag{20}$$

It is easy to verify that

$$L\begin{pmatrix} \lambda\psi_2^2 \\ -\lambda\psi_1^2 \end{pmatrix} = \lambda^2\begin{pmatrix} \lambda\psi_2^2 \\ -\lambda\psi_1^2 \end{pmatrix}. \tag{21}$$

We then take as our constraint the restriction of the variational derivatives of conserved quantities H_{2k_0} (for any fixed k_0) and λ_j [8, 10]:

$$\frac{\delta H_{2k_0}}{\delta u} - \beta\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = 0 \tag{22}$$

which in the present case implies

$$\begin{pmatrix} c_{2k_0+1} \\ b_{2k_0+1} \end{pmatrix} - \frac{1}{2}\beta\begin{pmatrix} \langle\Lambda\Psi_2, \Psi_2\rangle \\ -\langle\Lambda\Psi_1, \Psi_1\rangle \end{pmatrix} = 0. \tag{23}$$

(The constant β has been introduced for later convenience.) Hereafter we denote the inner product in $\mathbb{R}^N$ by $\langle ., .\rangle$ and

$$\Psi_1 = (\psi_{11}, \ldots, \psi_{1N})^{\mathrm{T}} \qquad \Psi_2 = (\psi_{21}, \ldots, \psi_{2N})^{\mathrm{T}} \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N). \tag{24}$$

It has been shown in [10] that (22) is invariant under all flows of (16). The system consisting of (18) and (22) is called a *constrained flow* and can be transformed into a finite-dimensional integrable Hamiltonian system (FDIHS) by introducing the so-called Jacobi–Ostrogradsky coordinates.

To deduce the Lax representation for the system (18) and (23) from the adjoint representation (8), we have to find the expressions of a_m, b_m, c_m under (18) and (23). Due to (12), (21) and (23), we may define the higher order terms [10] by

$$\begin{pmatrix} \tilde{c}_{2m+1} \\ \tilde{b}_{2m+1} \end{pmatrix} = \frac{1}{2}\beta \begin{pmatrix} \langle \Lambda^{2m-2k_0+1}\Psi_2, \Psi_2 \rangle \\ -\langle \Lambda^{2m-2k_0+1}\Psi_1, \Psi_1 \rangle \end{pmatrix} \qquad m \geqslant k_0 \tag{25}$$

and according to (10) and (18)

$$\tilde{a}_{2m} = -\frac{1}{q}\left(\tilde{b}_{2m+1} + \frac{1}{2}\tilde{b}_{2m-1,x}\right) = \frac{1}{2}\beta \langle \Lambda^{2m-2k_0}\Psi_1, \Psi_2 \rangle \qquad m > k_0. \tag{26}$$

By using equations (10), (12), (18) and (23), a direct calculation gives then expressions for the lower order terms a_{2m} for $m \leqslant k_0$ and b_{2m+1}, c_{2m+1} for $m < k_0$, which are denoted also by $\tilde{a}_{2m}, \tilde{b}_{2m+1}, \tilde{c}_{2m+1}$, respectively.

The construction of $\tilde{a}_m, \tilde{b}_m, \tilde{c}_m$ ensures that under (18) and (23)

$$\widetilde{V} = \sum_{m=0}^{\infty} \begin{pmatrix} \tilde{a}_m & \tilde{b}_m \\ \tilde{c}_m & -\tilde{a}_m \end{pmatrix} \lambda^{-m} \tag{27}$$

satisfies (8) as well. By direct calculation we find

$$\begin{aligned} M_{k_0} &= \begin{pmatrix} A^{(k_0)} & B^{(k_0)} \\ C^{(k_0)} & -A^{(k_0)} \end{pmatrix} \equiv \lambda^{2k_0}\widetilde{V} \\ A^{(k_0)} &= \sum_{m=0}^{k_0} \tilde{a}_{2m}\lambda^{2k_0-2m} + \frac{1}{2}\beta \sum_{j=1}^{N} \frac{\lambda_j^2 \psi_{1j}\psi_{2j}}{\lambda^2 - \lambda_j^2} \\ B^{(k_0)} &= \sum_{m=0}^{k_0-1} \tilde{b}_{2m+1}\lambda^{2k_0-2m-1} - \frac{1}{2}\beta \sum_{j=1}^{N} \frac{\lambda_j \lambda \psi_{1j}^2}{\lambda^2 - \lambda_j^2} \\ C^{(k_0)} &= \sum_{m=0}^{k_0-1} \tilde{c}_{2m+1}\lambda^{2k_0-2m-1} + \frac{1}{2}\beta \sum_{j=1}^{N} \frac{\lambda_j \lambda \psi_{2j}^2}{\lambda^2 - \lambda_j^2}. \end{aligned} \tag{28}$$

Since $\widetilde{V}$ under (18) and (23) satisfies (8), the M_{k_0} under (18) and (23) satisfies equation (8), too, namely

$$M_{k_0,x} = [U, M_{k_0}]. \tag{29}$$

Conversely, the construction of M_{k_0} guarantees that (29) is just the Lax representation for the system (18) and (23). This can also be verified by direct calculation.

We first present three systems of (18) and (23) below.

(a) When $k_0 = 0, \beta = 1$, (23) becomes

$$q = \tfrac{1}{2}\langle \Lambda\Psi_1, \Psi_1 \rangle \qquad r = -\tfrac{1}{2}\langle \Lambda\Psi_2, \Psi_2 \rangle \tag{30}$$

and then (18) can be written in canonical Hamiltonian form

$$\begin{aligned} \Psi_{1x} &= -\Lambda^2\Psi_1 + \frac{1}{2}\langle \Lambda\Psi_1, \Psi_1 \rangle \Lambda\Psi_2 = \frac{\partial \widetilde{H}_0}{\partial \Psi_2} \\ \Psi_{2x} &= -\frac{1}{2}\langle \Lambda\Psi_2, \Psi_2 \rangle \Lambda\Psi_1 + \Lambda^2\Psi_2 = -\frac{\partial \widetilde{H}_0}{\partial \Psi_1} \end{aligned} \tag{31}$$

$$\widetilde{H}_0 = -\langle\Lambda^2\Psi_1, \Psi_2\rangle + \tfrac{1}{4}\langle\Lambda\Psi_1, \Psi_1\rangle\langle\Lambda\Psi_2, \Psi_2\rangle. \tag{32}$$

The $A^{(0)}, B^{(0)}, C^{(0)}$ for M_0 in (28) read

$$\begin{aligned} A^{(0)}(\lambda) &= 1 + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j^2\psi_{1j}\psi_{2j}}{\lambda^2-\lambda_j^2} \\ B^{(0)}(\lambda) &= -\frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{1j}^2}{\lambda^2-\lambda_j^2} \\ C^{(0)}(\lambda) &= \frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{2j}^2}{\lambda^2-\lambda_j^2}. \end{aligned} \tag{33}$$

(b) When $k_0 = 1, \beta = \frac{1}{2}$, then (23) gives rise to the constraint

$$q_x = -q^2 r - \tfrac{1}{2}\langle\Lambda\Psi_1, \Psi_1\rangle \qquad r_x = r^2 q - \tfrac{1}{2}\langle\Lambda\Psi_2, \Psi_2\rangle \tag{34}$$

and by introducing

$$q_1 = q \qquad p_1 = r \tag{35}$$

the system (18) and (34) can be written in canonical Hamiltonian form

$$\Psi_{1x} = \frac{\partial\widetilde{H}_1}{\partial\Psi_2} \qquad q_{1x} = \frac{\partial\widetilde{H}_1}{\partial p_1} \qquad \Psi_{2x} = -\frac{\partial\widetilde{H}_1}{\partial\Psi_1} \qquad p_{1x} = -\frac{\partial\widetilde{H}_1}{\partial q_1} \tag{36}$$

$$\widetilde{H}_1 = -\tfrac{1}{2}q_1^2p_1^2 - \langle\Lambda^2\Psi_1, \Psi_2\rangle + \tfrac{1}{2}q_1\langle\Lambda\Psi_2, \Psi_2\rangle - \tfrac{1}{2}p_1\langle\Lambda\Psi_1, \Psi_1\rangle. \tag{37}$$

The $A^{(1)}, B^{(1)}, C^{(1)}$ for M_1 are of the form

$$\begin{aligned} A^{(1)}(\lambda) &= \lambda^2 - \frac{1}{2}q_1p_1 + \frac{1}{4}\sum_{j=1}^{N}\frac{\lambda_j^2\psi_{1j}\psi_{2j}}{\lambda^2-\lambda_j^2} \\ B^{(1)}(\lambda) &= -q_1\lambda - \frac{1}{4}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{1j}^2}{\lambda^2-\lambda_j^2} \\ C^{(1)}(\lambda) &= -p_1\lambda + \frac{1}{4}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{2j}^2}{\lambda^2-\lambda_j^2}. \end{aligned} \tag{38}$$

(c) When $k_0 = 2, \beta = 1$, (23) leads to the constraint

$$\begin{aligned} &\tfrac{1}{2}r_{xx} - \tfrac{3}{2}qrr_x + \tfrac{3}{4}q^2r^3 = -\langle\Lambda\Psi_2, \Psi_2\rangle \\ &\tfrac{1}{2}q_{xx} + \tfrac{3}{2}qrq_x + \tfrac{3}{4}q^3r^2 = \langle\Lambda\Psi_1, \Psi_1\rangle \end{aligned} \tag{39}$$

and by introducing the following Jacobi–Ostrogradsky coordinates:

$$q_1 = q \qquad q_2 = r \qquad p_1 = -\tfrac{3}{16}r^2q + \tfrac{1}{4}r_x \qquad p_2 = \tfrac{3}{16}q^2r + \tfrac{1}{4}q_x \tag{40}$$

the system (18) and (39) can be written in canonical Hamiltonian form

$$\Psi_{1x} = \frac{\partial\widetilde{H}_2}{\partial\Psi_2} \qquad q_{ix} = \frac{\partial\widetilde{H}_2}{\partial p_i} \qquad \Psi_{2x} = -\frac{\partial\widetilde{H}_2}{\partial\Psi_1} \qquad p_{ix} = -\frac{\partial\widetilde{H}_2}{\partial q_i} \tag{41}$$

where

$$\begin{aligned} \widetilde{H}_2 = {} & 4p_1p_2 - \tfrac{3}{4}q_1^2q_2p_1 + \tfrac{3}{4}q_2^2q_1p_2 - \tfrac{1}{64}q_1^3q_2^3 - \langle\Lambda^2\Psi_1, \Psi_2\rangle + \tfrac{1}{2}q_1\langle\Lambda\Psi_2, \Psi_2\rangle \\ & -\tfrac{1}{2}q_2\langle\Lambda\Psi_1, \Psi_1\rangle \end{aligned} \tag{42}$$

and the $A^{(2)}$, $B^{(2)}$, $C^{(2)}$ for M_2 are of the form

$$\begin{aligned}A^{(2)}(\lambda) &= \lambda^4 - \frac{1}{2}q_1q_2\lambda^2 + q_2p_2 - q_1p_1 + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j^2\psi_{1j}\psi_{2j}}{\lambda^2-\lambda_j^2}\\ B^{(2)}(\lambda) &= -q_1\lambda^3 + \left(\frac{1}{8}q_2q_1^2 + 2p_2\right)\lambda - \frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{1j}^2}{\lambda^2-\lambda_j^2}\\ C^{(2)}(\lambda) &= -q_2\lambda^3 + \left(\frac{1}{8}q_1q_2^2 - 2p_1\right)\lambda + \frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{2j}^2}{\lambda^2-\lambda_j^2}.\end{aligned} \tag{43}$$

2.3. Integrability

Denote M_0, M_1, or M_2 by M. It will be shown in the next section that M satisfies (1) with $r^{(12)}$ given by (54). An immediate consequence of (1) is that

$$\left\{\left(M^{(1)}(\lambda)\right)^2 \overset{\otimes}{,} \left(M^{(2)}(\mu)\right)^2\right\} = [\bar{r}_{12}(\lambda,\mu), M^{(1)}(\lambda)] - [\bar{r}_{21}(\mu,\lambda), M^{(2)}(\mu)] \tag{44}$$

where [2]

$$\bar{r}_{ij}(\lambda,\mu) = \sum_{k=0}^{1}\sum_{l=0}^{1}\left(M^{(1)}(\lambda)\right)^{1-k}\left(M^{(2)}(\mu)\right)^{1-l} r^{(ij)}(\lambda,\mu)\left(M^{(1)}(\lambda)\right)^{k}\left(M^{(2)}(\mu)\right)^{l}. \tag{45}$$

Then it follows immediately from (44) that

$$4\{\mathrm{Tr}\, M^2(\lambda), \mathrm{Tr}\, M^2(\mu)\} = \mathrm{Tr}\left\{\left(M^{(1)}(\lambda)\right)^2, \left(M^{(2)}(\mu)\right)^2\right\} = 0 \tag{46}$$

which ensures the involution property of the integrals of motion obtained from expanding M^2 in powers of λ.

For the system (31) one obtains

$$\mathrm{Tr}\, M^2(\lambda) = (A^{(0)}(\lambda))^2 + B^{(0)}(\lambda)C^{(0)}(\lambda) = 1 + \sum_{j=1}^{N}\frac{F_0^{(j)}}{\lambda^2-\lambda_j^2} \tag{47}$$

where

$$F_0^{(j)} = \lambda_j^2\psi_{1j}\psi_{2j} - \frac{1}{4}\langle\Lambda\Psi_1,\Psi_1\rangle\lambda_j\psi_{2j}^2 + \frac{1}{4}\sum_{k\neq j}\frac{\lambda_j\lambda_k}{\lambda_k^2-\lambda_j^2}(\lambda_j\psi_{1j}\psi_{2k} - \lambda_k\psi_{1k}\psi_{2j})^2 \qquad j = 1,\ldots,N. \tag{48}$$

and we have $\widetilde{H}_0 = -\sum_{i=1}^{N} F_0^{(j)}$.

For the system (36) we find

$$\mathrm{Tr}\, M^2(\lambda) = (A^{(1)}(\lambda))^2 + B^{(1)}(\lambda)C^{(1)}(\lambda) = \lambda^4 - 2\widetilde{H}_1 + \frac{1}{2}\sum_{j=1}^{N}\frac{F_1^{(j)}}{\lambda^2-\lambda_j^2} \tag{49}$$

where

$$\begin{aligned}F_1^{(j)} &= \lambda_j^4\psi_{1j}\psi_{2j} - \frac{1}{2}q_1\lambda_j^3\psi_{2j}^2 + \frac{1}{2}p_1\lambda_j^3\psi_{1j}^2 - \frac{1}{8}\langle\Lambda\Psi_1,\Psi_1\rangle\lambda_j\psi_{2j}^2\\ &\quad + \frac{1}{8}\sum_{k\neq j}\frac{\lambda_j\lambda_k}{\lambda_k^2-\lambda_j^2}(\lambda_j\psi_{1j}\psi_{2k} - \lambda_k\psi_{1k}\psi_{2j})^2 \qquad j = 1,\ldots,N.\end{aligned} \tag{50}$$

For the system (41) one gets

$$\mathrm{Tr}\, M^2(\lambda) = (A^{(2)}(\lambda))^2 + B^{(2)}(\lambda)C^{(2)}(\lambda) = \lambda^8 - \widetilde{H}_2\lambda^2 + F_2^{(0)} + \sum_{j=1}^{N} \frac{F_2^{(j)}}{\lambda^2 - \lambda_j^2} \tag{51}$$

where

$$\begin{aligned} F_2^{(0)} = {} & \langle \Lambda^4\Psi_1, \Psi_2\rangle - \tfrac{1}{2}q_1q_2\langle\Lambda^2\Psi_1, \Psi_2\rangle - \tfrac{1}{2}q_1\langle\Lambda^3\Psi_2, \Psi_2\rangle \\ & + \tfrac{1}{2}q_2\langle\Lambda^3\Psi_1, \Psi_1\rangle + (p_2q_2 - q_1p_1)^2 + (p_2 + \tfrac{1}{16}q_1^2q_2)\langle\Lambda\Psi_2, \Psi_2\rangle \\ & + (p_1 - \tfrac{1}{16}q_2^2q_1)\langle\Lambda\Psi_1, \Psi_1\rangle \end{aligned}$$

$$\begin{aligned} F_2^{(j)} = {} & \left(\lambda_j^4 - \frac{1}{2}q_1q_2\lambda_j^2 + p_2q_2 - q_1p_1\right)\lambda_j^2\psi_{1j}\psi_{2j} + \left(p_2 - \frac{1}{2}q_1\lambda_j^2 + \frac{1}{16}q_1^2q_2\right)\lambda_j^3\psi_{2j}^2 \\ & + \left(p_1 + \frac{1}{2}q_2\lambda_j^2 - \frac{1}{16}q_2^2q_1\right)\lambda_j^3\psi_{1j}^2 - \frac{1}{4}\langle\Lambda\Psi_1, \Psi_1\rangle\lambda_j\psi_{2j}^2 \\ & + \frac{1}{4}\sum_{k\neq j}\frac{\lambda_j\lambda_k}{\lambda_k^2 - \lambda_j^2}(\lambda_j\psi_{1j}\psi_{2k} - \lambda_k\psi_{1k}\psi_{2j})^2 \qquad j = 1, \ldots, N. \end{aligned} \tag{52}$$

Then equation (46) and, for example, (51) guarantees that the functionally independent integrals of motion $\widetilde{H}_2$ and $F^{(j)}$, $j = 0, 1, \ldots, N$, are in involution. This shows the integrability of (31), (36) and (41) in the Liouville sense [16].

3. The (non)dynamical r-matrices

3.1. The Kaup–Newell system

We start the presentation of our results by first displaying some non-dynamical r-matrices, obtained by the method discussed above.

The classical Poisson structure associated with the Lax representation discussed in the previous section, equations (31), (36) and (41) is the following. With respect to the standard Poisson bracket, it is found by direct calculation that both $A^{(0)}$, $B^{(0)}$, $C^{(0)}$ and $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ as well as $A^{(2)}$, $B^{(2)}$, $C^{(2)}$ satisfy the following relations:

$$\begin{aligned} &\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0 \\ &\{A(\lambda), B(\mu)\} = \frac{\beta\mu}{\mu^2 - \lambda^2}(\mu B(\mu) - \lambda B(\lambda)) \\ &\{A(\lambda), C(\mu)\} = \frac{\beta\mu}{\mu^2 - \lambda^2}(\lambda C(\lambda) - \mu C(\mu)) \\ &\{B(\lambda), C(\mu)\} = \frac{2\beta\lambda\mu}{\mu^2 - \lambda^2}(A(\mu) - A(\lambda)). \end{aligned} \tag{53}$$

For the elementary Poisson brackets (53) one finds that (1) holds with

$$r^{(ij)} = \frac{\beta\alpha_i\alpha_j}{\alpha_j^2 - \alpha_i^2}P^{(ij)} - \frac{\beta\alpha_i}{\alpha_j + \alpha_i}S^{(ij)} \qquad S^{(ij)} = \frac{1}{2}(\sigma_0^{(i)} \otimes \sigma_0^{(j)} + \sigma_3^{(i)} \otimes \sigma_3^{(j)}). \tag{54}$$

(Here the σ_i's are the standard Pauli matrices, and the permutation matrix P is given by $P^{(ij)} = \frac{1}{2}\sum_{n=0}^{3}\sigma_n^{(i)} \otimes \sigma_n^{(j)}$.) In fact, equations (1), (54) hold for all FDIHS obtained from the constrained flows (18) and (23). The classical Poisson structure (1), (54) contains all necessary information of the present system, and is more rich than the Lax representation [1].

The r-matrix in (54) is of the 'twisted two-pole' structure in the nomenclature of [17]: if we define

$$R(\lambda,\mu) := -\beta\mu\left\{\frac{\mu S}{\lambda^2-\mu^2}+\frac{\lambda(P-S)}{\lambda^2-\mu^2}\right\} \tag{55}$$

then $r^{(ij)} = R(\alpha_i,\alpha_j) - R(\alpha_i,\infty)$.

3.2. *The KdV hierarchy*

For the KdV hierarchy [18], the eigenvalue problem is of the form

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(q,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad U(q,\lambda)=\begin{pmatrix}0 & 1\\ -\lambda-q & 0\end{pmatrix} \tag{56}$$

the second constrained flow with constraint $q=\frac{1}{8}\langle\Psi_1,\Psi_1\rangle$ reads [8]

$$\Psi_{1x}=\Psi_2=\frac{\partial\widetilde{H}}{\partial\Psi_2} \qquad \Psi_{2x}=-\frac{1}{8}\langle\Psi_1,\Psi_1\rangle\Psi_1-\Lambda\Psi_1=-\frac{\partial\widetilde{H}}{\partial\Psi_1} \tag{57}$$

with the Hamiltonian

$$\widetilde{H}=\tfrac{1}{2}\langle\Lambda\Psi_1,\Psi_1\rangle+\tfrac{1}{2}\langle\Psi_2,\Psi_2\rangle+\tfrac{1}{32}\langle\Psi_1,\Psi_1\rangle^2. \tag{58}$$

The Lax representation for (57) is given by (29)

$$\begin{aligned}M(\lambda) &\equiv \begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & -A(\lambda)\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ -\lambda-\frac{1}{16}\langle\Psi_1,\Psi_1\rangle & 0\end{pmatrix}+\frac{1}{16}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}\psi_{1j}\psi_{2j} & -\psi_{1j}^2\\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j}\end{pmatrix}.\end{aligned} \tag{59}$$

and we have

$$\begin{aligned}&\{A(\lambda),A(\mu)\}=\{B(\lambda),B(\mu)\}=0\\ &\{C(\lambda),C(\mu)\}=\tfrac{1}{4}(A(\lambda)-A(\mu))\\ &\{A(\lambda),B(\mu)\}=\frac{1}{8(\mu-\lambda)}(B(\mu)-B(\lambda))\\ &\{A(\lambda),C(\mu)\}=\frac{1}{8(\mu-\lambda)}(C(\lambda)-C(\mu))-\frac{1}{8}B(\lambda)\\ &\{B(\lambda),C(\mu)\}=\frac{1}{4(\mu-\lambda)}(A(\mu)-A(\lambda)).\end{aligned} \tag{60}$$

Then equations (60) give rise to the classical Poisson structure (1) for the system (57) (in fact for all constrained flows of KdV hierarchy) with the r-matrix given by

$$r^{(ij)}(\alpha_i,\alpha_j)=\frac{1}{8(\alpha_j-\alpha_i)}P^{(ij)}+\frac{1}{8}S^{(ij)} \qquad S^{(ij)}=\sigma_-^{(i)}\otimes\sigma_-^{(j)} \tag{61}$$

and this r satisfies the classical Yang–Baxter equations of the form $r^{(ijk)}=0$ (5). (In this case $r^{(ij)}\neq -r^{(ji)}$ and the index order in (5) is important.) (This r-matrix is related to the one presented in [19] (equation (27)) by a singular limit.)

3.3. The TG hierarchy

The above examples have an r-matrix that depended only on the spectral parameters. We will now present two constrained flows that lead to a dynamical r-matrix.

Let us first consider the TG hierarchy associated with the following eigenvalue problem [20]:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(u,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad U(u,\lambda) = \begin{pmatrix} -\lambda + \frac{1}{2}q & r \\ r & \lambda - \frac{1}{2}q \end{pmatrix}. \tag{62}$$

By using the method in [8, 10], the first constrained flow, with constraint $q = (\langle\Psi_2, \Psi_2\rangle - \langle\Psi_1, \Psi_1\rangle)/G$, $r = 2G$ is found as follows:

$$\begin{aligned} \Psi_{1x} &= -\Lambda\Psi_1 + \frac{1}{2G}(\langle\Psi_2, \Psi_2\rangle - \langle\Psi_1, \Psi_1\rangle)\Psi_1 + 2G\Psi_2 = \frac{\partial \widetilde{H}}{\partial \Psi_2} \\ \Psi_{2x} &= 2G\Psi_1 + \Lambda\Psi_2 - \frac{1}{2G}(\langle\Psi_2, \Psi_2\rangle - \langle\Psi_1, \Psi_1\rangle)\Psi_2 = -\frac{\partial \widetilde{H}}{\partial \Psi_1} \end{aligned} \tag{63}$$

where

$$\widetilde{H} = -\langle\Lambda\Psi_1, \Psi_2\rangle + G(\langle\Psi_2, \Psi_2\rangle - \langle\Psi_1, \Psi_1\rangle) \qquad G = \sqrt{\langle\Psi_1, \Psi_2\rangle}. \tag{64}$$

Using the method of [9], we obtain the Lax representation for (63) given by (29) with

$$\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\lambda & G \\ G & \frac{1}{2}\lambda \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j}\begin{pmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} \end{pmatrix}. \tag{65}$$

One gets

$$\begin{aligned} &\{A(\lambda), A(\mu)\} = 0 \\ &\{B(\lambda), B(\mu)\} = \frac{1}{G}(B(\lambda) - B(\mu)) \\ &\{C(\lambda), C(\mu)\} = -\frac{1}{G}[C(\lambda) - C(\mu)] \\ &\{A(\lambda), B(\mu)\} = \frac{2}{\mu - \lambda}(B(\mu) - B(\lambda)) \\ &\{A(\lambda), C(\mu)\} = \frac{2}{\mu - \lambda}(C(\lambda) - C(\mu)) \\ &\{B(\lambda), C(\mu)\} = \frac{4}{\mu - \lambda}(A(\mu) - A(\lambda)) + \frac{1}{G}(B(\lambda) + C(\mu)) \end{aligned} \tag{66}$$

which gives rise to the classical Poisson structure (1) for the system (63) with the dynamical r-matrix given by

$$r^{(ij)}(\alpha_i, \alpha_j) = \frac{2}{\alpha_j - \alpha_i}P^{(ij)} + \frac{1}{2G}S^{(ij)} \qquad S^{(ij)} = \sigma_3^{(i)} \otimes \sigma_1^{(j)}. \tag{67}$$

This satisfies the classical, dynamical Yang–Baxter equations (4), (6) with

$$X^{(ijk)} = -\frac{1}{2G^3}\sigma_3^{(i)} \otimes \sigma_3^{(j)} \otimes \sigma_1^{(k)}. \tag{68}$$

In this case the dynamical variables appear only through G.

3.4. The Wadati–Konno–Ichikawa hierarchy

Finally consider the Wadati–Konno–Ichikawa (WKI) hierarchy associated with the following eigenvalue problem [21]:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(u,\lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad U(u,\lambda) = \begin{pmatrix} \lambda & \lambda q \\ \lambda r & -\lambda \end{pmatrix}. \tag{69}$$

By using the method in [8–10], we obtain the first constrained flow, with the constraint $q = -\langle \Lambda\Psi_1, \Psi_1\rangle/G$, $r = \langle \Lambda\Psi_2, \Psi_2\rangle/G$ as follows:

$$\begin{aligned} \Psi_{1x} &= \Lambda\Psi_1 - \frac{1}{G}\langle \Lambda\Psi_1, \Psi_1\rangle\Lambda\Psi_2 = \frac{\partial \widetilde{H}}{\partial \Psi_2} \\ \Psi_{2x} &= \frac{1}{G}\langle \Lambda\Psi_2, \Psi_2\rangle\Lambda\Psi_1 - \Lambda\Psi_2 = -\frac{\partial \widetilde{H}}{\partial \Psi_1} \end{aligned} \tag{70}$$

where

$$\widetilde{H} = \langle \Lambda\Psi_1, \Psi_2\rangle - G \qquad G = \sqrt{1 + \langle \Lambda\Psi_1, \Psi_1\rangle\langle \Lambda\Psi_2, \Psi_2\rangle}. \tag{71}$$

The Lax representation for (70) is given by (29) with

$$\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix} + \frac{1}{2}\sum_{j=1}^{N} \frac{\lambda_j}{\lambda - \lambda_j}\begin{pmatrix} \lambda_j\psi_{1j}\psi_{2j} & -\lambda\psi_{1j}^2 \\ \lambda\psi_{2j}^2 & -\lambda_j\psi_{1j}\psi_{2j} \end{pmatrix} \tag{72}$$

and we have

$$\begin{aligned} &\{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0 \\ &\{A(\lambda), A(\mu)\} = \frac{1}{2G}\langle \Lambda\Psi_2, \Psi_2\rangle(\lambda B(\lambda) - \mu B(\mu)) + \frac{1}{2G}\langle \Lambda\Psi_1, \Psi_1\rangle(\lambda C(\lambda) - \mu C(\mu)) \\ &\{A(\lambda), B(\mu)\} = \frac{\lambda\mu}{\lambda - \mu}B(\lambda) - \frac{\mu^2}{\lambda - \mu}B(\mu) + \frac{1}{G}\langle \Lambda\Psi_1, \Psi_1\rangle\mu A(\mu) \\ &\{A(\lambda), C(\mu)\} = \frac{\lambda\mu}{\mu - \lambda}C(\lambda) - \frac{\mu^2}{\mu - \lambda}C(\mu) + \frac{1}{G}\langle \Lambda\Psi_2, \Psi_2\rangle\mu A(\mu) \\ &\{B(\lambda), C(\mu)\} = \frac{2\lambda\mu}{\lambda - \mu}(A(\lambda) - A(\mu)). \end{aligned} \tag{73}$$

This leads to the classical Poisson structure (1) for (70) with the r-matrix given by

$$\begin{aligned} &r^{(ij)}(\alpha_i, \alpha_j) = \frac{\alpha_i\alpha_j}{\alpha_j - \alpha_i}P^{(ij)} - \alpha_i S^{(ij)} + \frac{\alpha_i}{2G}E^{(ij)} \\ &S^{(ij)} = \tfrac{1}{2}(\sigma_0^{(i)} \otimes \sigma_0^{(j)} + \sigma_3^{(i)} \otimes \sigma_3^{(j)}) \qquad E^{(ij)} = F^{(i)} \otimes \sigma_3^{(j)} \\ &F^{(i)} = \langle \Lambda\Psi_1, \Psi_1\rangle\sigma_+^{(i)} - \langle \Lambda\Psi_2, \Psi_2\rangle\sigma_-^{(i)}. \end{aligned} \tag{74}$$

This r-matrix satisfies the dynamical classical Yang–Baxter equation (4), (6), with

$$X^{(ijk)} = -\frac{\alpha_i\alpha_j}{2G^3}[F^{(i)} \otimes F^{(j)} \otimes \sigma_3^{(k)} + 2\sigma_+^{(i)} \otimes \sigma_-^{(j)} \otimes \sigma_3^{(k)} + 2\sigma_-^{(i)} \otimes \sigma_+^{(j)} \otimes \sigma_3^{(k)}]. \tag{75}$$

Now the dynamical variables appear through both $\langle \Lambda\Psi_1, \Psi_1\rangle$ and $\langle \Lambda\Psi_2, \Psi_2\rangle$ as well as G.

4. Conclusions

In this paper we have discussed the classical Poisson structure and the related (dynamical) r-matrix for some finite-dimensional integrable Hamiltonian systems. These integrable systems were derived by constraining the integrable flow of an evolution equation in a particular way [8–10, 22]. This seems to be a fruitful way of producing dynamical systems with interesting types of r-matrices.

The possibility of a *dynamical* r-matrices has been known for some time now, but no general theory for such systems has been developed so far. It is therefore important to derive examples using different methods, in order to find what the essential features are. For example, one may ask in what way the Jacobi identities are satisfied. The examples presented in section 3 belong to the class that satisfy them only through the most general form proposed so far, equation (6). The method presented in this paper can probably be used to generate still other types of interesting examples.

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References

[1] Faddeev L D and Takhtajan L A 1986 *Hamiltonian Method in the Theory of Solitons* (Berlin: Springer)
[2] Babelon O and Viallet C-M 1990 *Phys. Lett.* **237B** 411
[3] Maillet J M 1985 *Phys. Lett.* **162B** 137
[4] Semenov-Tian-Shanskii M 1983 *Funct. Anal. Appl.* **17** 259
[5] Eilbeck J C, Enolskii V Z, Kusnetsov V B and Tsiganov A V 1994 *J. Phys. A: Math. Gen.* **27** 567
[6] Sklyanin E K 1994 *Algebra i Analiz* **6** 227
[7] Avan J and Talon M 1993 *Phys. Lett.* **303B** 33
[8] Zeng Y B 1991 *Phys. Lett.* **160A** 541
[9] Zeng Y B 1993 *J. Phys. A: Math. Gen.* **26** L273
[10] Zeng Y B 1994 *Physica* **73D** 171
[11] Antonowicz M and Rauch-Wojciechowski S 1993 *Inverse Problems* **9** 201
[12] Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
[13] Tu G Z 1989 *J. Math. Phys.* **30** 330
[14] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia, PA: SIAM)
[15] Tu G Z 1993 Gradients and coadjoint representations of isospectral problem *Preprint*
[16] Arnold V I 1978 *Mathematical Method of Classical Mechanics* (Berlin: Springer)
[17] Avan J and Talon M 1991 *Nucl. Phys.* B **352** 215
[18] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transformation* (Philadelphia, PA: SIAM)
[19] Avan J 1991 *Phys. Lett.* **156A** 61
[20] Tu G Z and Meng D Z 1989 *Acta Math. Appl. Sinica* **5** 89
[21] Wadati M, Konno K and Ichikawa Y H 1979 *J. Phys. Soc. Japan* **47** 1698
[22] Zeng Y B 1993 *Chinese Sci. Bull.* **38** 1161